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SIMPLIFICATION OF COMBINATORIAL LINK FLOER HOMOLOGY

ANNA BELIAKOVA

ABSTRACT. We define a new combinatorial complex computing the hat version of link Floer homology over $\mathbb{Z}/2\mathbb{Z}$, which turns out to be significantly smaller than the Manolescu–Ozsváth–Sarkar one.

INTRODUCTION

Knot Floer homology is a powerful knot invariant constructed by Ozsváth–Szabó [13] and Rasmussen [16]. In its basic form, the knot Floer homology $\widehat{\mathrm{HFK}}(K)$ of a knot $K \in S^3$ is a finite-dimensional bigraded vector space over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$

$$\widehat{\mathrm{HFK}}(K) = \bigoplus_{d \in \mathbb{Z}, i \in \mathbb{Z}} \widehat{\mathrm{HFK}}_d(K, i),$$

where d is the Maslov and i is the Alexander grading. Its graded Euler characteristic

$$\sum_{d, i} (-1)^d \mathrm{rank} \widehat{\mathrm{HFK}}_d(K, i) t^i = \Delta_K(t)$$

is equal to the symmetrized Alexander polynomial $\Delta_K(t)$. The knot Floer homology enjoys the following symmetry extending that of the Alexander polynomial.

$$(1) \quad \widehat{\mathrm{HFK}}_d(K, i) = \widehat{\mathrm{HFK}}_{d-2i}(K, -i)$$

By the result of Ozsváth–Szabó [12], the maximal Alexander grading i , such that $\widehat{\mathrm{HFK}}_*(K, i) \neq 0$ is the Seifert genus $g(K)$ of K . Moreover, Ghiggini showed for $g(K) = 1$ [4] and Yi Ni in general [9], that the knot is fibered if and only if $\mathrm{rank} \widehat{\mathrm{HFK}}_*(K, g(K)) = 1$. A concordance invariant bounding from below the slice genus of the knot can also be extracted from knot Floer homology [11]. For torus knots the bound is sharp, providing a new proof of the Milnor conjecture. The first purely combinatorial proof of the Milnor conjecture was given by Rasmussen in [17] by using Khovanov homology [5].

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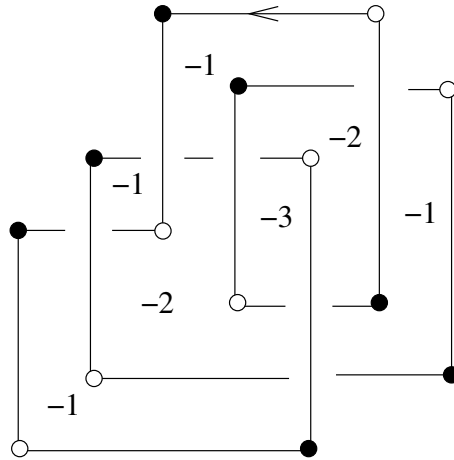


FIGURE 1. **A rectangular diagram for 5_2 knot.** The number associated to a domain is minus the winding number for its points. The sets \mathbb{X} and \mathbb{O} consist of black and white points, respectively.

Knot Floer homology was extended to links in [15]. The first combinatorial construction of the link Floer homology was given in [7] over \mathbb{F} and then in [8] over \mathbb{Z} . Both constructions use grid diagrams of links.

A *grid diagram* is a square grid on the plane with $n \times n$ squares. Each square is decorated either with an X , an O , or nothing. Moreover, every row and every column contains exactly one X and one O . The number n is called *complexity* of the diagram. Following [8], we denote the set of all O 's and X 's by \mathbb{O} and \mathbb{X} , respectively.

Given a grid diagram, we construct an oriented, planar link projection by drawing horizontal segments from the O 's to the X 's in each row, and vertical segments from the X 's to the O 's in each column. We assume that at every intersection point the vertical segment overpasses the horizontal one. This produces a planar *rectangular diagram* G for an oriented link L in S^3 . Any link in S^3 admits a rectangular diagram (see e.g. [3]). An example is shown in Figure 1.

In [7], [8] the grid lies on the torus, obtained by gluing the top most segment of the grid to the bottommost one and the leftmost segment to the right most one. In the torus, the horizontal and vertical segments of the grid become circles. The MOS complex is then generated by n -tuples of intersection points between horizontal and vertical circles, such that exactly one point belongs to each horizontal (or vertical) circle. Two generators \mathbf{x} and \mathbf{y} are connected by the differential if $n - 2$ points of \mathbf{x} and \mathbf{y} coincide and if there exists a rectangle with vertices among \mathbf{x} and \mathbf{y} and edges lying alternatively on horizontal and vertical circles, which does not contain

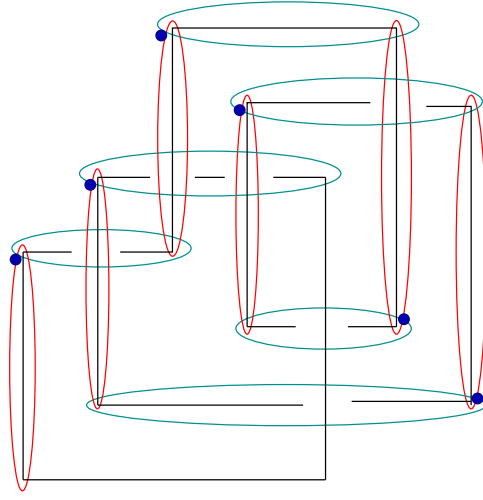


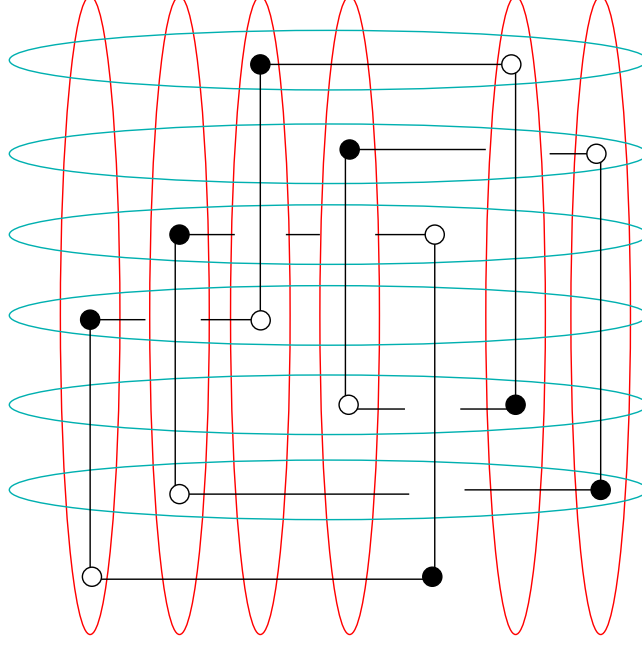
FIGURE 2. **Collection of short ovals for 5_2 knot.** The dots show a generator in Alexander grading 1.

X 's and O 's and points among \mathbf{x} and \mathbf{y} . The Alexander grading is given by formula (2) below, and the Maslov grading by (3) plus one. The MOS complex has $n!$ generators. This number greatly exceeds the rank of its homology. For the trefoil, for example, the number of generators is 120, while the rank of $\widehat{\text{HFK}}(3_1)$ is 3.

In this paper, we construct another combinatorial complex computing link Floer homology, which has significantly less generators. All knots with less than 6 crossings admit rectangular diagrams where all differentials in our complex are zero, and the rank of the homology group is equal to the number of generators.

Main results. Our construction also uses rectangular diagrams. Given an oriented link L in S^3 , let G be its rectangular diagram in \mathbb{R}^2 . Let us draw $2n - 2$ narrow short ovals around all but one horizontal and all but one vertical segments of the rectangular diagram G in such a way, that the outside domain has at least one X or O . We denote by \mathbf{S} the set of unordered $(n - 1)$ -tuples of intersection points between the horizontal and vertical ovals, such that exactly one point belongs to each horizontal (or vertical) oval. We assume throughout this paper that the ovals intersect transversally. An example is shown in Figure 2.

A chain complex $(C(G), \partial)$ computing the hat version of link Floer homology of L over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ is defined as follows. The generators are elements of \mathbf{S} . The bigrading on \mathbf{S} can be constructed analogously to those in [7]. Suppose ℓ is the number of components of L . Then the Alexander grading is a function $A: \mathbf{S} \rightarrow (\frac{1}{2}\mathbb{Z})^\ell$, defined as follows.

FIGURE 3. **The complex $(C(G), \partial)$ with long ovals.**

First, we define a function $a : \mathbf{S} \rightarrow \mathbb{Z}^\ell$. For a point p , the i^{th} component of a is minus the winding number of the projection of the i^{th} component of the oriented link around p . In the grid diagram, we have $2n$ distinguished squares containing X 's or O 's. Let $\{c_{i,j}\}$, $i \in \{1, \dots, 2n\}$, $j \in \{1, \dots, 4\}$, be the vertices of these squares. Given $\mathbf{x} \in \mathbf{S}$, we set

$$(2) \quad A(\mathbf{x}) = \sum_{x \in \mathbf{x}} a(x) - \frac{1}{8} \left(\sum_{i,j} a(c_{i,j}) \right) - \left(\frac{n_1 - 1}{2}, \dots, \frac{n_\ell - 1}{2} \right),$$

where here n_i is the complexity of the i^{th} component of L , i.e. the number of horizontal segments belonging to this component.

The homological or Maslov grading is a function $M : \mathbf{S} \rightarrow \mathbb{Z}$ defined as follows. Given two collections A, B of finitely many points in the plane, let $\mathcal{I}(A, B)$ be the number of pairs $(a_1, a_2) \in A$ and $(b_1, b_2) \in B$ with $a_1 < b_1$ and $a_2 < b_2$. Define

$$(3) \quad M(\mathbf{x}) = \mathcal{I}(\mathbf{x}, \mathbf{x}) - \mathcal{I}(\mathbf{x}, \mathbb{O}) - \mathcal{I}(\mathbb{O}, \mathbf{x}) + \mathcal{I}(\mathbb{O}, \mathbb{O}).$$

The differential in our complex is defined as follows. We first consider the link Floer homology complex $(C(S^2, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbb{X}, \mathbb{O}), \partial)$ of Ozsváth–Szabó with extra base-points \mathbb{X} and \mathbb{O} (the definition is recalled in Section 1). We assume that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $2(n-1)$ long ovals (as long as $n \times n$ grid) encircling all but one horizontal

and vertical segments of G in such a way, that the outside region contains at least one point among \mathbb{X} or \mathbb{O} (compare Figure 3). Like the MOS complex, the resulting complex is combinatorial. Two generators \mathbf{x} and \mathbf{y} are connected by the differential if they differ by two or one points and if there exists a rectangle or a bigon, with vertices among \mathbf{x} and \mathbf{y} and edges lying alternatively on horizontal and vertical ovals, which does not contain X 's and O 's and points among \mathbf{x} and \mathbf{y} inside.

To this complex we further apply a simple lemma from homological algebra, which allows us to reduce the set of generators to \mathbf{S} (or to short the ovals α and β) keeping track of the differential. In Section 3, we describe an algorithm which for any domain connecting two generators decides whether it counts for the differential or not. Furthermore, we distinguish a large class of domains which always count. In general, however the count depends on the order of shortening of ovals, which replace in some sense the choice of a complex structure in the analytic setting.

Let V_i be the two dimensional bigraded vector space over \mathbb{F} spanned by one generator in Alexander and Maslov gradings zero and another one in Maslov grading -1 and Alexander grading minus the i -th basis vector.

Theorem 1. *Suppose G is a rectangular diagram of an oriented ℓ component link L , where the i^{th} component of L has complexity n_i . Then the homology $H_*(C(G), \partial)$ is equal to $\widehat{\text{HFL}}(L) \otimes \bigotimes_{i=1}^{\ell} V_i^{n_i-1}$.*

The complex $(C(G), \partial)$ has much fewer generators than the MOS complex (compare Section 4). A computer program realizing our algorithm was written by Droz [2]. It allows to determine the Seifert genus and fibereness of knots at least until 18 crossings. Recently, Droz introduced signs in our construction [1]. He is actually modifying his program to compute knot Floer homology over \mathbb{Z} . This will allow to study the torsion in this homology theory.

The paper is organized as follows. We recall the construction of link Floer homology in Section 1. Theorem 1 is proven in Section 2. In Section 3 we introduce a big class of domains which always count for the differential. In the last section we compute Floer homology of 5_2 knot and discuss further computations made by Droz.

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1. LINK FLOER HOMOLOGY WITH MULTIPLE BASEPOINTS

For the readers convenience, we review here the Ozsváth–Szabó construction of knot and link Floer homologies, considering the case where the link meets the Heegaard surface in extra intersection points. Our exposition follows closely [7, Section 2].

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ be a Heegaard diagram for S^3 , where Σ is a surface of genus g , k is some positive integer, $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_{g+k-1}\}$ are pairwise disjoint, embedded curves in Σ which span a half-dimensional subspace of $H_1(\Sigma; \mathbb{Z})$ (and hence specify a handlebody U_α with boundary equal to Σ), $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_{g+k-1}\}$ is another collection of attaching circles specifying U_β , and $\mathbf{w} = \{w_1, \dots, w_k\}$ and $\mathbf{z} = \{z_1, \dots, z_k\}$ are distinct marked points with

$$\mathbf{w}, \mathbf{z} \subset \Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}.$$

Let $\{A_i\}_{i=1}^k$ resp. $\{B_i\}_{i=1}^k$ be the connected components of $\Sigma - \boldsymbol{\alpha}$ resp. $\Sigma - \boldsymbol{\beta}$.

We suppose that our basepoints are placed in such a manner that each component A_i or B_i contains exactly one basepoint among the \mathbf{w} and exactly one basepoint among the \mathbf{z} . We can label our basepoints so that A_i contains z_i and w_i , and then B_i contains w_i and $z_{\nu(i)}$, for some permutation ν of $\{1, \dots, k\}$.

In this case, the basepoints uniquely specify an oriented link L in $S^3 = U_\alpha \cup U_\beta$, by the following conventions. For each $i = 1, \dots, k$, let ξ_i denote the arc in A_i from z_i to w_i and let η_i denote the arc in B_i from w_i to $z_{\nu(i)}$. Let $\tilde{\xi}_i \subset U_\alpha$ be an arc obtained by pushing the interior of ξ_i into U_α , and $\tilde{\eta}_i$ be the arc obtained by pushing the interior of η_i into U_β . Now, we can let L be the oriented link obtained as the union

$$\bigcup_{i=1}^k (\tilde{\xi}_i + \tilde{\eta}_i).$$

Definition 1.1. In the above case, we say that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is a $2k$ -pointed Heegaard diagram compatible with the oriented link L in S^3 .

Let ℓ denote the number of components of L . Clearly, $k \geq \ell$. In the case where $k = \ell$, these are the Heegaard diagrams used in the definition of link Floer homology [15], see also [13], [16]. In the case where $k > \ell$, these Heegaard diagrams can still be used to calculate link Floer homology, in a suitable sense.

Definition 1.2. A *periodic domain* is a two-chain of the form

$$P = \sum_{i=1}^k (a_i \cdot A_i + b_i \cdot B_i)$$

such that $w_i \notin P$ for all i . A Heegaard diagram is said to be *admissible* if for every non-trivial periodic domain, the set $\{a_i, b_i\}$ contains some strictly positive and negative numbers.

For simplicity, we consider now the case where L is a knot. The case of links can be handled with a few minor notational changes in Subsection 1.1 below.

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ be a Heegaard diagram compatible with an oriented knot K . Let us consider the complex $C(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ generated over \mathbb{F} by intersection points between tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_{g+k-1}$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_{g+k-1}$ in $\text{Sym}^{g+k-1}(\Sigma)$ endowed with the differential

$$(4) \quad \partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\left\{ \phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid \begin{array}{l} M(\phi) = 1, \\ n_{w_i}(\phi) = n_{z_i}(\phi) = 0 \ \forall i = 1, \dots, k \end{array} \right\}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}} \right) \cdot \mathbf{y}.$$

where $\pi_2(\mathbf{x}, \mathbf{y})$ denotes the space of homology classes of Whitney disks (domains) connecting \mathbf{x} to \mathbf{y} , $M(\phi)$ is the Maslov index of ϕ , $n_p(\phi)$ denotes the local multiplicity of ϕ at the reference point p (i.e. the algebraic intersection number of ϕ with the subvariety $\{p\} \times \text{Sym}^{g+k-2}(\Sigma)$), $\mathcal{M}(\phi)$ is the moduli space of pseudo-holomorphic representatives of ϕ , and $\#()$ denotes a count modulo two. In the case when the Heegaard diagram is admissible, the sum in Equation (4) is finite. We refer to [14] for further details.

The *relative Alexander grading* of two intersection points \mathbf{x} and \mathbf{y} is defined by the formula

$$A(\mathbf{x}) - A(\mathbf{y}) = \left(\sum_{i=1}^k n_{z_i}(\phi) \right) - \left(\sum_{i=1}^k n_{w_i}(\phi) \right).$$

The absolute A -grading can be fixed by requiring

$$\sum_{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} t^{A(\mathbf{x})} \equiv \Delta_K(t) \cdot (1 - t^{-1})^{k-1} \pmod{2},$$

where $\Delta_K(t)$ is the symmetrized Alexander polynomial which could be made to work over \mathbb{Z} by introducing signs.

Moreover, there is a *relative Maslov grading*, defined by

$$(5) \quad M(\mathbf{x}) - M(\mathbf{y}) = M(\phi) - 2 \sum_{i=1}^k n_{w_i}(\phi).$$

The relative Maslov grading can also be lifted to an absolute grading as explained in e.g. [7].

For this complex, the function A defines an Alexander grading which is preserved by the differential. The following proposition shows how to extract the usual knot Floer homology from the above variants using multiple basepoints. The proof is given in [7], compare also [15].

Proposition 1.1. *Let $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ be a $2k$ -pointed admissible Heegaard diagram compatible with a knot K . Then, we have an identification*

$$(6) \quad H_*(C(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z}), \partial) \cong \widehat{\mathrm{HFK}}(K) \otimes V^{\otimes(k-1)},$$

where V is the two-dimensional vector space spanned by two generators, one in bigrading $(-1, -1)$, another in bigrading $(0, 0)$.

1.1. Modifications for links. Recall that knot Floer homology has a generalization to the case of oriented links L . For an ℓ component oriented link L in S^3 , this takes the form of a multi-graded theory

$$\widehat{\mathrm{HFL}}(L) = \bigoplus_{d \in \mathbb{Z}, h \in \mathbb{H}} \widehat{\mathrm{HFL}}_d(L, h),$$

where $\mathbb{H} \cong H_1(S^3 - L) \cong \mathbb{Z}^\ell$, with the latter isomorphism induced by an ordering of the link components. The Euler characteristic determines the multivariable Alexander polynomial $\Delta_L(t_1, \dots, t_\ell)$ as follows.

$$\chi(\widehat{\mathrm{HFL}}(L)) = \begin{cases} \prod_{i=1}^{\ell} (t_i^{1/2} - t_i^{-1/2}) & \Delta_L(t_1, \dots, t_\ell) & \ell > 1 \\ \Delta_L(t) & & \ell = 1 \end{cases}$$

We sketch now the changes to be made to the above discussion to define link Floer homology for Heegaard diagrams with extra basepoints.

Suppose now that $(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ is a Heegaard diagram compatible with an oriented link L in the sense of Definition 1.1.

Let us label the basepoints keeping track of which link component they belong to. Specifically, suppose L is a link with ℓ components, and for $i = 1, \dots, \ell$, n_i is the number of pairs of basepoints on the i^{th} component. Letting S be the index set of pairs (i, j) with $i = 1, \dots, \ell$ and $j = 1, \dots, n_i$. We now have basepoints $\{z_{i,j}\}_{(i,j) \in S}$ and $\{w_{i,j}\}_{(i,j) \in S}$.

We can now form the chain complex $C(\Sigma, \alpha, \beta, \mathbf{w}, \mathbf{z})$ over \mathbb{F} analogous to the version before, generated by intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. This complex has a relative Maslov grading as before. It also has an Alexander grading which in this

case is an ℓ -tuple of integers,

$$A: \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \left(\frac{1}{2}\mathbb{Z}\right)^\ell,$$

determined up to an overall additive constant by the formula

$$A(\mathbf{x}) - A(\mathbf{y}) = \left(\sum_{j=1}^{n_1} (n_{z_{1,j}}(\phi) - n_{w_{1,j}}(\phi)), \dots, \sum_{j=1}^{n_\ell} (n_{z_{\ell,j}}(\phi) - n_{w_{\ell,j}}(\phi)) \right).$$

The indeterminacy in this case can be fixed with the help of Proposition 1.2.

The differential drops Maslov grading by one and preserves the Alexander multi-grading, and hence the homology groups $H_*(C(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z}))$ inherit a Maslov grading and an Alexander multi-grading.

Proposition 1.2. *Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ be a $2k$ -pointed admissible Heegaard diagram compatible with an oriented link L , with n_i pairs of basepoints corresponding to the i^{th} component of L . Then, there are multi-graded identifications*

$$H_*(C(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z}), \partial) \cong \widehat{\text{HFL}}(L) \otimes \bigotimes_{i=1}^{\ell} V_i^{\otimes (n_i-1)},$$

where V_i is the two-dimensional vector space spanned by one generator in Maslov and Alexander gradings zero, and another in Maslov grading -1 and Alexander grading corresponding to minus the i^{th} basis vector.

By the result of [15], the bigraded groups $\widehat{\text{HFL}}(L)$ are link invariants. In particular, they do not change if

- the complex structure on Σ is varied;
- the $\boldsymbol{\alpha}$ - and $\boldsymbol{\beta}$ -curves are moved by isotopies (in the complement of the basepoints);
- the $\boldsymbol{\alpha}$ - and $\boldsymbol{\beta}$ -curves are moved by handle-slides (in the complement of the basepoints);
- the Heegaard diagram is stabilized.

1.2. MOS complex. It was shown in [7], that for a link of complexity n the MOS complex, described in Introduction, coincides with the complex $(C(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z}), \partial)$, where Σ is a torus, $\boldsymbol{\alpha}$ are n standard meridians on Σ , $\boldsymbol{\beta}$ are n standard longitudes (both providing $n \times n$ grid on the torus), and where the basepoints \mathbf{w} and \mathbf{z} are identified with \mathbb{X} and \mathbb{O} . Hence, by Proposition 1.2 the homology of this complex is equal to $\widehat{\text{HFL}}(L) \otimes \bigotimes_{i=1}^{\ell} V_i^{\otimes (n_i-1)}$.

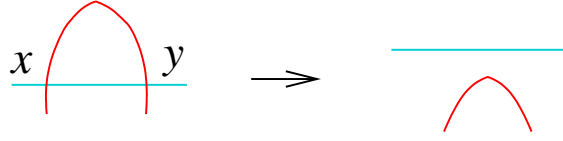


FIGURE 4. Removing of a bigon without basepoints inside.

2. THE COMPLEX $(C(G), \partial)$

2.1. Shortening of ovals. Suppose G is a rectangular diagram of complexity n for an oriented link L . Let (C, ∂) be the complex $(C(S^2, \alpha, \beta, \mathbb{X}, \mathbb{O}), \partial)$, where α and β are $2(n - 1)$ ovals encircling all but one horizontal and vertical segments of G , respectively. It is easy to check that any periodic domain in this case has positive and negative coefficients. The differential ∂ is given by counting of all Maslov index one domains connecting two generators, which do not contain X 's and O 's and admit holomorphic representatives.

The next lemma allows to “short” an oval by removing any bigon without basepoints inside (see Figure 4).

Lemma 2.1. *Assume that α and β curves form a bigon with corners x and y without basepoints inside. Then the complex (C, ∂) is homotopy equivalent to the complex (C', ∂') whose set of generators $\mathbf{S}(C')$ is obtained from $\mathbf{S}(C)$ by removing all generators containing x or y .*

Proof. Let us define the homotopy equivalence explicitly.

Let $I : \mathbf{S}(C') \rightarrow \mathbf{S}(C)$ and $P : \mathbf{S}(C) \rightarrow \mathbf{S}(C')$ be the obvious inclusion and projection. For any $\mathbf{x} \in \mathbf{S}(C)$, let $h(\mathbf{x})$ be zero if $y \notin \mathbf{x}$, otherwise $h(\mathbf{x})$ is obtained from \mathbf{x} by replacing y by x . We define $F : \mathbf{S}(C) \rightarrow \mathbf{S}(C')$ and $G : \mathbf{S}(C') \rightarrow \mathbf{S}(C)$ as follows.

$$F = P(\text{Id} + \partial h) \quad G = (\text{Id} + h\partial)I$$

Here Id is the identity map. Then $GF - \text{Id} = \partial h + h\partial$, i.e. GF is homotopic to the identity on C . On the other hand, FG is the identity map on C' .

Let us endow the complex C' with the differential $\partial' = F\partial G$. It is easy to check that $\partial'^2 = 0$ and that F and G are chain maps. Indeed, $\partial'F = F\partial IP$ and $IP\partial G = G\partial'$, or $\partial'F = F\partial$ and $\partial G = G\partial'$ over \mathbb{F} .

Let us describe the new differential $\partial' = P(\partial + \partial h\partial)I$ in more details. Assume $\mathbf{y} \in \partial\mathbf{x}$ and \mathbf{x}, \mathbf{y} do not contain the corners of the bigon x and y . For all such \mathbf{x} and \mathbf{y} , ∂ and ∂' coincide, i.e. we have $\mathbf{y} \in \partial'\mathbf{x}$. Furthermore, assume $\mathbf{a}, \mathbf{b} \in \mathbf{S}(C)$ do not contain x and y , then for any $\mathbf{x}, \mathbf{y} \in \mathbf{S}(C)$ with $y \in \mathbf{y}$, $x \in \mathbf{x}$, such that $h(\mathbf{y}) = \mathbf{x}$, $\mathbf{y} \in \partial\mathbf{a}$, and $\mathbf{b} \in \partial\mathbf{x}$, \mathbf{b} occurs once in $\partial'\mathbf{a}$. \square

2.2. Definition of the complex $(C(G), \partial)$. We define the complex $(C(G), \partial)$ as obtained from the complex with long ovals by applying Lemma 2.1 several times, until the complex has \mathbf{S} as the set of generators. This subsection aims to give a recursive definition of the differential ∂ in this complex. In general, it will depend on the order in which the ovals were shortened.

Given $\mathbf{x}, \mathbf{y} \in \mathbf{S}$, there is an oriented closed curve $\gamma_{\mathbf{x}, \mathbf{y}}$ composed of arcs belonging to horizontal and vertical ovals, where each piece of a horizontal oval connects a point in \mathbf{x} to a point in \mathbf{y} (and hence each piece of the vertical one goes from a point in \mathbf{y} to a point in \mathbf{x}). In S^2 , there exists an oriented (immersed) domain $D_{\mathbf{x}, \mathbf{y}}$ bounded by $\gamma_{\mathbf{x}, \mathbf{y}}$. The points \mathbf{x} and \mathbf{y} are called *corners* of the domain $D_{\mathbf{x}, \mathbf{y}}$.

Let D_i be the closures of the connected components of the complement of ovals in S^2 . Suppose that the orientation of D_i is induced by the orientation of S^2 . Then we say that a domain $D = \sum_i n_i D_i$ connects two generators if for all i , $n_i \geq 0$ and D is connected. Let \mathfrak{D} be the set of all domains connecting two generators whose Maslov indices (given by (3)) differ by one and which do not contain \mathbb{X} and \mathbb{O} inside.

We define

$$\partial \mathbf{x} := \sum_{\mathbf{y}} \sum_{D_{\mathbf{x}, \mathbf{y}} \in \mathfrak{D}} m(D_{\mathbf{x}, \mathbf{y}}) \mathbf{y},$$

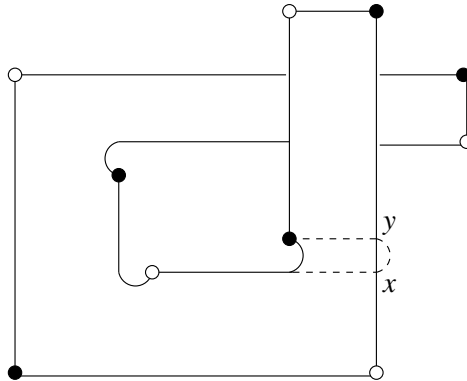
where $m(D_{\mathbf{x}, \mathbf{y}}) \in \{0, 1\}$ is a multiplicity of the domain. In what follows we give an algorithm which determines these multiplicities.

Let us fix the order in which the ovals were shortened. For any domain $D_{\mathbf{a}, \mathbf{b}} \in \mathfrak{D}$ we prolongate the last oval which was shortened to obtain this domain, and show whether in the resulting complex $(C'(G), \partial')$, one can find \mathbf{x} and \mathbf{y} with $\mathbf{y} \in \partial' \mathbf{a}$, $h(\mathbf{y}) = \mathbf{x}$ and $\mathbf{b} \in \partial' \mathbf{x}$ as in the proof of Lemma 2.1. If there is an odd number of such \mathbf{x} and \mathbf{y} , then $m(D_{\mathbf{a}, \mathbf{b}}) = 1$, otherwise the multiplicity is zero. To determine $m(D_{\mathbf{a}, \mathbf{y}})$ and $m(D_{\mathbf{x}, \mathbf{b}})$ in the new complex, we prolongate the next oval, and continue to do so until the domains in question are polygons which always count.

This process is illustrated in Figure 5, where \mathbf{a} and \mathbf{b} are given by black and white points respectively; \mathbf{y} is obtained from \mathbf{a} by switching the black point on the dashed oval to y and the upper black point to the white point on the same oval; \mathbf{x} is obtained from \mathbf{y} by switching y to x . Here the algorithm terminates after the first step.

In fact, the algorithm terminates already when the domains in question are strongly indecomposable with nondegenerate system of cuts (as defined in the next section) since they all count for the differential.

2.3. Proof of Theorem 1. By Lemma 2.1, the complex $(C(G), \partial)$ is homotopy equivalent to the complex with long ovals. Moreover, the Alexander and Maslov



gradings are fixed in such a way, that

This can be shown by comparing our and MOS complexes. Indeed, one can construct a bijection identifying the bigraded generators of our and MOS complexes which do not cancel on the level of the Euler characteristic (see [1] for more details).

3. INDECOMPOSABLE DOMAINS

3.1. Maslov index. Let $e(D)$ be the Euler measure of the domain D , which for any surface S with k acute right-angled corners, l obtuse ones, and Euler characteristic $\chi(S)$ is equal to $\chi(S) - k/4 + l/4$. Moreover, the Euler measure is additive under disjoint unions and gluings along boundaries. In [6, Section 4], Lipshitz give a formula computing the Maslov index $M(D_{\mathbf{x}, \mathbf{y}})$ of $D_{\mathbf{x}, \mathbf{y}}$ as follows.

where $n_{\mathbf{x}} = \sum_{x \in \mathbf{x}} n_x$. The number n_x is the local multiplicity of the domain at the corner x , e.g. $n_x = 0$ for an isolated corner, $n_x = 1/4$ for an acute (or $\pi/2$ -angled) corner, $n_x = 1/2$ for a straight (or π -angled) corner or $n_x = 3/4$ for an obtuse (or $3\pi/2$ -angled) one. For a composition of two domains $D_{\mathbf{x},\mathbf{z}} = D_{\mathbf{x},\mathbf{y}} \cup D_{\mathbf{y},\mathbf{z}}$, we have $M(D_{\mathbf{x},\mathbf{z}}) = M(D_{\mathbf{x},\mathbf{y}}) + M(D_{\mathbf{y},\mathbf{z}})$.

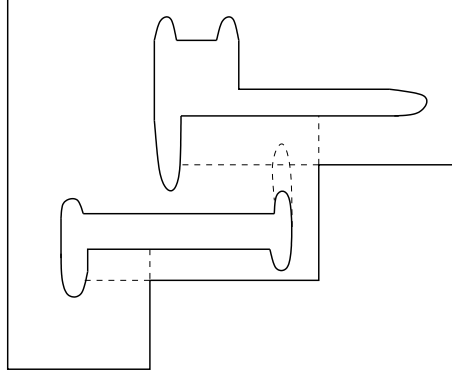


FIGURE 6. **Indecomposable, but not strongly indecomposable domain.**

The oval destroying the indecomposability is shown by the dashed line.

A domain D is called *decomposable* if it is a composition of Maslov index zero and one domains; otherwise, the domain is indecomposable. A domain is called *strongly indecomposable* if it is indecomposable and no prolongations of ovals inside this domain destroy its indecomposability. An example of an indecomposable, but not strongly indecomposable domain is shown in Figure 6.

3.2. Count of domains. In what follows any domain is assumed to belong to \mathfrak{D} , i.e. to have Maslov index one and no basepoints inside, and to connect two generators. We will also assume that our domains have no corners with negative multiplicities or with multiplicities bigger than $3/4$, since such domains never count (which can easily be seen by applying the algorithm from the previous section).

A path in a domain starting at an obtuse or straight corner and following a horizontal or vertical oval until the boundary of the domain will be called a *cut*. There are two cuts at any obtuse corner and one at any straight corner.

We say that a cut touch a boundary component A at an oval B if either the end point of this cut belongs to B or the cut is a prolongation of B . We define the distance between two cuts touching A to be odd, if one of them touches A at a vertical oval and another one at a horizontal oval; otherwise the distance is even.

A system of cuts in a domain is called *degenerate* if there is a boundary component such that all cuts touching this component have an even distance to each other; otherwise, the system of cuts is called *nondegenerate*. An example of a degenerate system of cuts is given in Figure 7. Since two cuts (a horizontal and a vertical one) leave any obtuse corner, a system of cuts can be degenerate only if our domain has at least one inner boundary component without obtuse corners. Let us call such an inner boundary component *bad*.

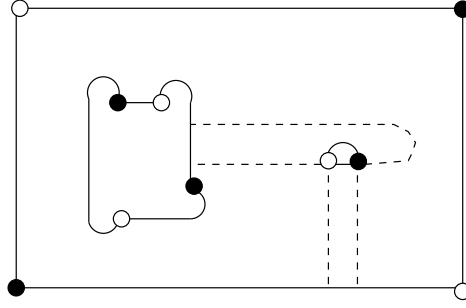


FIGURE 7. An indecomposable domain with a degenerate system of cuts.

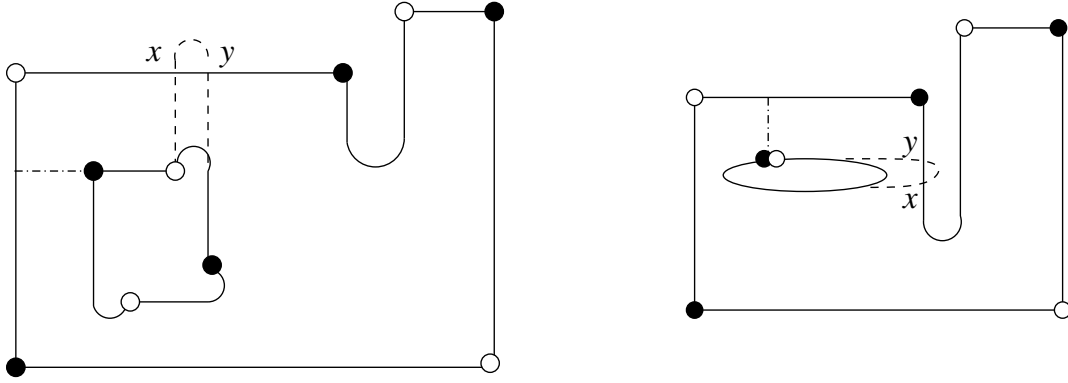


FIGURE 8. Strongly indecomposable domains realizing differentials from black to white points. The prolonged oval is shown by a dashed line.

Theorem 3.1. *Any strongly indecomposable domain with a nondegenerate system of cuts counts for the differential.*

Corollary 3.2. *Any strongly indecomposable domain without bad components counts for the differential.*

Proof. The proof is by induction on the number of boundary components c in the domain. If $c = 1$, it is easy to see by prolongating ovals that any immersed polygon counts.

Assume that for $c = n - 1$, the claim holds. Suppose our complex has a strongly indecomposable domain $D \in \mathfrak{D}$ with a nondegenerate system of cuts and n boundary components. Examples with $c = 2$ are drawn in Figure 8.

Let us stretch one oval in D . Then D gives rise to a domain D' with $c = n - 1$ boundary components. (Prolongations splitting one boundary component into two

are considered as immersions.) Let x and y be the corners of the bigon, obtained after stretching. The stretched oval connects y to some boundary component, say A . By Lemma 3.4 below, y can also be connected with A by a unique path inside D . This path do not intersect the prolonged part, since our domain is strongly indecomposable. Hence, D' can be represented as a union of two domains connecting some generators. The unique path connecting y with A leaves any boundary component along a cut. Moreover, if the path goes along the boundary of a component without obtuse corners, then it also meets a component with two obtuse corners (compare Proof of Lemma 3.4). Therefore, D' is a union of two strongly indecomposable Maslov index one domains with nondegenerate system of cuts, which count for the differential by the induction hypothesis. Applying Lemma 2.1, we conclude that the domain D also counts for the differential. \square

3.3. Structure of domains. Here we provide some technical results needed for the proof of Theorem 3.1.

Definition 3.3. A boundary component $C_1 \subset \partial D_{\mathbf{x},\mathbf{y}}$ is called y -connected with another component $C_2 \subset \partial D_{\mathbf{x},\mathbf{y}}$ if for any point $y \in C_1$ disjoint from the corners, there exists a unique path starting at y and ending in C_2 , such that

- 1) the path goes along cuts or $\partial D_{\mathbf{x},\mathbf{y}}$, where the segments of horizontal and vertical ovals alternate along the path;
- 2) the corners of the path (i.e. intersection points of horizontal and vertical segments) come alternatively from \mathbf{x} and $\tilde{\mathbf{y}}$, where $\tilde{\mathbf{y}}$ contains \mathbf{y} and intersection points of cuts with $\partial D_{\mathbf{x},\mathbf{y}}$;
- 3) the first corner belongs to \mathbf{x} .

Lemma 3.4. *In an indecomposable domain with a nondegenerate system of cuts, any two boundary components are y -connected.*

Proof. Recall that an inner boundary component without obtuse corners is called bad. Let b be the number of bad components and c be the total number of boundary components in our domain. Assume first that the domain has no straight corners. In this case, we prove the claim by induction on c and b .

Assume $b = 0$, $c = 2$. If one of the cuts goes from the inner boundary component to itself, the domain is decomposable (see Figure 9 (a)). If it is not the case, then an easy check verifies the claim (compare Figure 9 (b)).

Assume the claim holds for $b = 0$, $c = n - 1$. Let us add an n -th good component A to the domain. Then if there are no cuts ending at A , we are done, since two

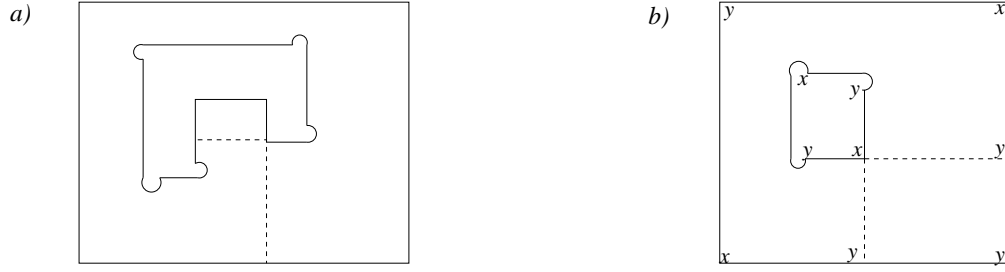


FIGURE 9. **Case** $b = 0$, $c = 2$. *a)* Decomposable domain. The cuts are shown by dashed lines. *b)* Indecomposable domain. The corners from \mathbf{x} are marked by x and the points from $\tilde{\mathbf{y}}$ by y .

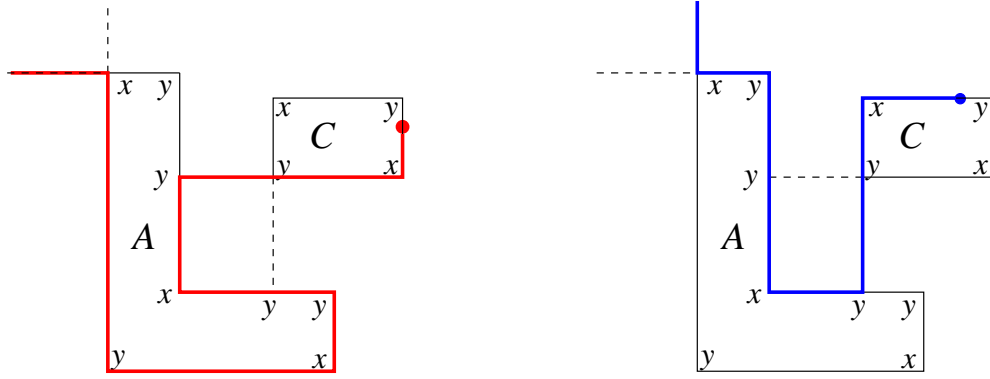


FIGURE 10. C is y -connected to A . The two choices of y are shown by red and blue dots. The connected paths have the corresponding colors. All corners without cuts are assumed to be acute.

cuts from the obtuse corner y -connect A to any other component by the induction hypothesis.

If there is a component connected with A by two cuts, then it is y -connected with the outside exactly in the case when A has this property. To check this, it is sufficient to find a required path for two choices of y (before and after one corner) on this component. An example is shown in Figure 10. Note that a particular form of the domain does not matter for the argument.

If all cuts ending at A come from components connected with A by two cuts (as in Figure 10), then A is y -connected to the outside by the previous argument. Suppose there is a component $C \subset \partial D_{\mathbf{x}, \mathbf{y}}$ connected with A by just one cut (see Figure 11). Then there are two possibilities: either the path described in Definition 3.3 after leaving A (along one of the cuts) comes back through C without visiting all other

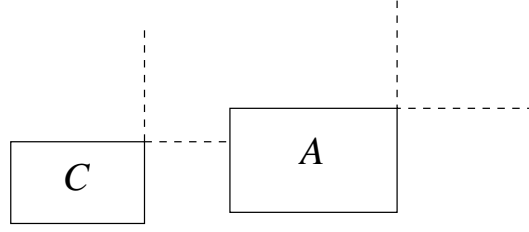


FIGURE 11. C is connected to A by one cut. All corners without cuts are assumed to be acute.

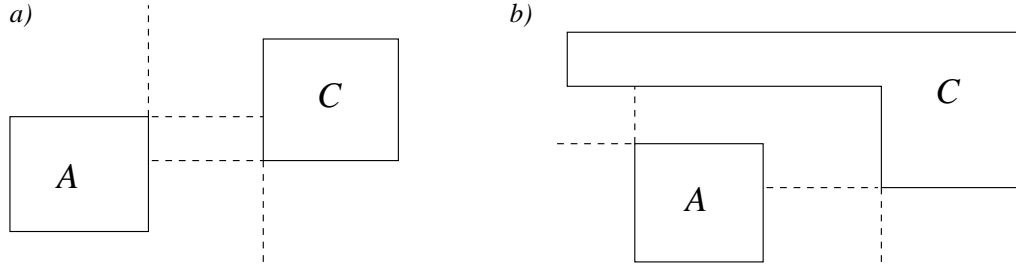


FIGURE 12. **Case** $b = 0$, $c = n$. a) Decomposable domain. b) Indecomposable domain, the system of cuts is nondegenerate.

components or it does not happens. In the first case, the domain is decomposable, since the path can be used to split the domain. In the second case, all components are y -connected.

The case when C and A exchange two cuts is similar: either the domain is decomposable or all components are y -connected (compare Figure 12). If C and A exchange three or more cuts, the domain is decomposable.

Assume $b = 1$, $c = n$. Let us denote by B the bad component. Then, since $M(D_{\mathbf{x},\mathbf{y}}) = 1$, either the outer boundary component has an obtuse corner or an inner boundary component $E \subset \partial D_{\mathbf{x},\mathbf{y}}$ has two obtuse corners. In the first case, both cuts from the obtuse corner on the outer component should end at B , otherwise the system of cuts is degenerate or the domain is decomposable.

Let us consider the second case. If B is not connected with $\partial D_{\mathbf{x},\mathbf{y}}$ by a cut, the domain is decomposable. If there are three cuts connecting E with B or connecting E with $\partial D_{\mathbf{x},\mathbf{y}} \setminus B$, then the domain is decomposable again.

If there are two paths connecting E with B (or with $\partial D_{\mathbf{x},\mathbf{y}} \setminus B$), such that each of them leaves E at a different obtuse corner, then the domain is either decomposable or have a degenerate system of cuts.

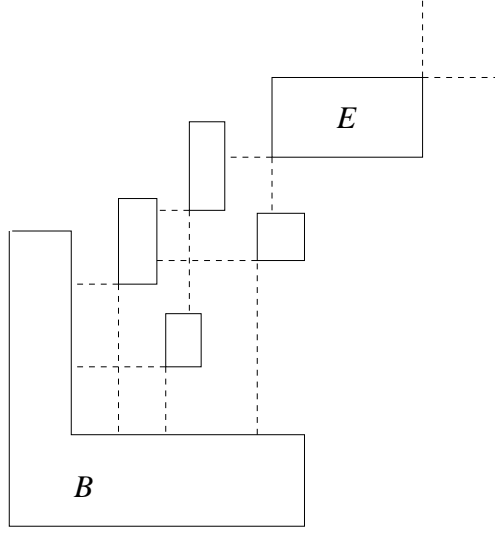


FIGURE 13. **Case** $b = 1$, $c = n$. All drawn components are y -connected. All corners without cuts are assumed to be acute.

Hence, in our case, two cuts from one obtuse corner of E should connect E with components, say A_1, \dots, A_k , which are otherwise not connected with B , and two cuts from the other obtuse corner should connect E with B . Therefore, the situation looks like in Figure 13, where all components are y -connected.

If there are components connected with B by one or two cuts, they are also y -connected with the remaining components, since B is y -connected with them.

The case $b = k$, $c = n$ is similar. Again $2k$ cuts should be absorbed by bad components. If a bad component is touched by no pair of cuts at odd distance, the system of cuts is degenerate; otherwise the component is y -connected to the outside or the domain is decomposable.

It remains to consider the case where $D_{\mathbf{x}, \mathbf{y}}$ has straight corners. In this case, one of the boundary components is an oval with a common corner of two generators. Let us call such a component special. An example is shown in Figure 8. Let us first assume that we have only one special component. Then the cut from the straight corner y -connects this component with any other one (otherwise not connected with the special one) by the previous argument. If we have a component connected with the special one by one or two cuts, then it is also y -connected with all the others, since the special component has this property. The obvious induction completes the proof. Note that if two (or more) special components exchange their cuts, then the domain is decomposable. \square

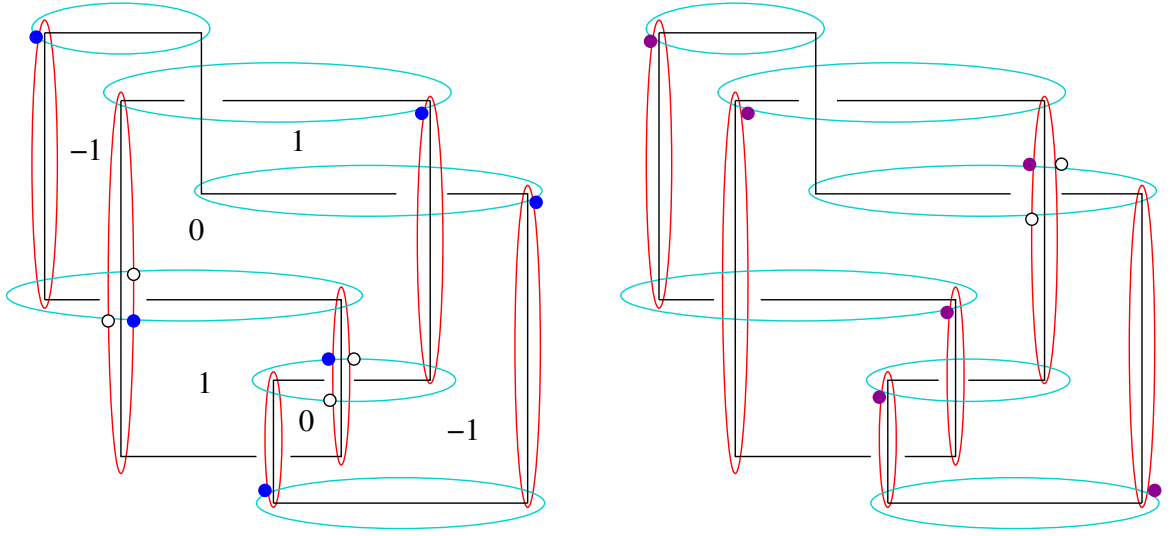


FIGURE 14. **A complex for 5_2 knot.** The colored dots show generators in the maximal Alexander grading equal to 1. A number assigned to a region is minus the winding number for its points.

4. COMPUTATIONS

In this section we show how $\widehat{\text{HFK}}$ of small knots can be computed by hands and discuss the computer program written by Droz.

4.1. **5_2 knot.** Figure 14 shows a rectangular diagram for 5_2 knot of complexity $n = 7$ obtained from the original diagram in Figure 1 by cyclic permutations (compare [3]). An advantage of this diagram is that there are no regions counted for the differential.

The Alexander grading of a generator is given by the formula $A(\mathbf{x}) = \sum_{x \in \mathbf{x}} a(x) - 2$. The maximal Alexander grading is equal to one. There are two generators in this grading shown by colored dots in Figure 14. Both of them have Maslov grading 2.

The homology of our complex is $\widehat{\text{HFK}}(5_2) \otimes V^6$. Hence in Alexander grading zero, we have 12 additional generators coming from the multiplication with V . Note that our complex has 15 generators in Alexander grading zero. Indeed, 12 of them can be obtained by moving one point of a generator in Alexander grading one to the other side of the oval. In three cases, depicted by white dots there are two possibilities to move a point. This gives 3 additional generators. Note that all movings drop Maslov index by one. We deduce that $\widehat{\text{HFK}}(5_2, 0)$ has rank three. To compute $\widehat{\text{HFK}}$ in the negative Alexander gradings we use the symmetry (1).

Finally, we derive that $\widehat{\text{HFK}}(5_2)$ has rank two in the Alexander–Maslov bigrading $(1, 2)$, rank three in $(0, 1)$, and rank two in the bigrading $(-1, 0)$. To compare, the

Alexander polynomial is $\Delta_{5_2}(t) = 2(t + t^{-1}) - 3$. The knot 5_2 is not fibered and its Seifert genus is one.

4.2. Droz's program. Droz wrote a computer program calculating the homology of our complex, which is already installed on the Bar–Natan's Knot Atlas [2]. As a byproduct, his program generates rectangular diagrams of knots and links and allows to change them by Cromwell–Dynnikov moves. The program can be used to determine Seifert genus and fibereness of knots at least until 18 crossings. The precise description of the algorithm is given in [1], where Droz also introduces signs in our construction.

According to Droz's computations, the number of generators in our complex is significantly smaller than that in the MOS complex. Moreover, for small knots, almost all domains suitable for the differential are embedded polygons, so they always count for the differential. For example, for knots admitting rectangular diagrams of complexity 10, the number of generators in the MOS complex is $10! = 3'628'800$. Our complex has on average about 50'000 generators among them about 1'000 in the positive Alexander gradings. The knot $12n2000$ admits a rectangular diagram of complexity 12, where $12! = 479'001'600$. Our complex has 1'411'072 generators with 16'065 in the positive Alexander gradings.

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